Edge states

Consider the problem of electrons confined in a Hall bar of finite width by a non-uniform electric field. For simplicity, we will consider the situation where the potential $V(x)$ is smooth on the scale of the magnetic length, but this is not central to the discussion. If we assume that the system still has translation symmetry in the y direction, the solution to the Schrödinger equation must still be of the form

$$\psi(x,y) = \frac{1}{\sqrt{L_y}} e^{iky} f_k(x)$$

The function f_k will no longer be a simple harmonic wave function as we found in the case of the uniform electric field. However we can anticipate that f_k will still be peaked near (but in general not precisely at) the point $X_k = -kl_B^2$. The eigenvalues ε_k will no longer be precisely linear in k but will still reflect the kinetic energy of the cyclotron motion plus the local potential energy $V(X_k)$. The group velocity

$$\vec{v}_k = \frac{1}{\hbar}\frac{\partial \epsilon_k}{\partial k}\hat{y}$$

has the opposite sign on the two edges of the sample. This means that in the ground state there are edge currents of opposite sign flowing in the sample. The semi-classical interpretation of these currents is that they represent 'skipping orbits' in which the circular cyclotron motion is interrupted by collisions with the walls at the edges.

To calculate the current we have to add up the group velocities of all the occupied states

$$I = -\frac{e}{L_y}\int_{-\infty}^{\infty} dk \frac{L_y}{2\pi}\frac{1}{\hbar}\frac{\partial \epsilon_k}{\partial k} n_k$$

where for the moment we assume that in the bulk, only a single Landau level is occupied and n_k is the probability that state $k$in that Landau level is occupied. Assuming zero temperature and noting that the integrand is a perfect derivative, we have

$$I = -\frac{e}{h}\int_{\mu_R}^{\mu_L} d\epsilon = -\frac{e}{h}[\mu_R - \mu_L]$$

(To understand the order of limits of integration, recall that as k increases, X_k decreases.) The definition of the Hall voltage drop is

$$eV_H = e[V_R - V_L] = [\mu_R - \mu_L]$$

(To get the signs straight here, note that an increase in chemical potential brings in more electrons. This is equivalent to a more positive voltage and hence a more negative potential energy –eV . Since H – μN enters the thermodynamics, electrostatic potential energy and

chemical potential move the electron density oppositely. V and μ thus have the same sign of effect because electrons are negatively charged.)

Hence

$$I = -\nu \frac{e^2}{h} V_H$$

where we have now allowed for the possibility that ν different Landau levels are occupied in the bulk and hence there are ν separate edge channels contributing to the current. Using the above result and the fact that the current flows at right angles to the voltage drop we have the desired results

$$\sigma_{xx} = 0 \qquad \sigma_{xy} = -\nu \frac{e^2}{h}$$

with the quantum number ν being an integer.

So far we have been ignoring the possible effects of disorder. Remarkably, in the QHE case, the back scattering is essentially zero in very wide samples. To see this note that in the case of the Hall bar, scattering into a backward moving state would require transfer of the electron from one edge of the sample to the other since the edge states are spatially separated. For samples which are very wide compared to the magnetic length (more precisely, to the Anderson localization length) the matrix element for this is exponentially small. If the disorder causes Landau level mixing at the edges to occur (because the confining potential is relatively steep) then it is possible for an electron in one edge channel to scatter into another, but the current is still going in the same direction so that there is no reduction in overall transmission probability. It is this chiral (unidirectional) nature of the edge states which is responsible for the fact that the Hall conductance is correctly quantized independent of the disorder.

Disorder will broaden the Landau levels in the bulk and provide a reservoir of (localized) states which will allow the chemical potential to vary smoothly with density. These localized states will not contribute to the transport and so the Hall conductance will be quantized over a plateau of finite width in B (or density) as seen in the data. Thus obtaining the universal value of quantized Hall conductance to a precision of 10^{-10} does not require fine tuning the applied B field to a similar precision.