Landau levels in 2D

For the vector potential one convenient choice is the so-called Landau gauge:

$$\vec{A}(\vec{r}) = xB\hat{y}$$

Which obeys $\vec{\nabla} \times \vec{A} = B\hat{z}$. In this gauge the vector potential points in the y direction but varies only with the x position. Notice that the magnetic field (and hence all the physics) is translationally invariant, but the Hamiltonian is not.

The Hamiltonian can be written in the Landau gauge as

$$H = \frac{1}{2m}\left(p_x^2 + \left(p_y + eBx\right)^2\right)$$

Taking advantage of the translation symmetry in the y direction, let us attempt a separation of variables by writing the wave function in the form

$$\psi_k(x, y) = e^{iky} f_k(x)$$

This has the advantage that it is an eigenstate of p_y and hence we can make the replacement $p_y \to \hbar k$ in the Hamiltonian. After separating variables we have the effective one-dimensional Schrödinger equation

$$h_k f_k(x) = \epsilon_k f_k(x)$$

where

$$h_k = \frac{1}{2m} p_x^2 + \frac{1}{2m}(\hbar k + eBx)^2$$

This is simply a one-dimensional displaced harmonic oscillator

$$h_k = \frac{1}{2m} p_x^2 + \frac{1}{2} m\omega_c^2 (x + k l_B^2)^2$$

whose frequency is the classical cyclotron frequency ($\omega_c = \frac{eB}{m}$) and whose central position $X_k = -k l_B^2$ is (somewhat paradoxically) determined by the y momentum quantum number ($l_B^2 = \frac{\hbar}{eB}$). Thus for each plane wave chosen for the y direction there will be an entire family of energy eigenvalues

$$\epsilon_{kn} = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

which depend only on n and are completely independent of the y momentum $\hbar k$. The corresponding (unnormalized) eigenfunctions are

$$\psi_{nk}(\vec{r}) = \frac{1}{\sqrt{L}} e^{iky} H_n(x - X_k) e^{-\frac{1}{2l_B^2}(x-X_k)^2}$$

where H_n is (as usual for harmonic oscillators) the nth Hermite polynomial (in this case displaced to the new central position X_k).

These harmonic oscillator levels are called Landau levels. Due to the lack of dependence of the energy on k, the degeneracy of each level is enormous.

We assume periodic boundary conditions in the y direction. Because of the vector potential, it is impossible to simultaneously have periodic boundary conditions in the x direction. However since the basis wave functions are harmonic oscillator polynomials multiplied by strongly converging gaussians, they rapidly vanish for positions away from the center position $X_k = -kl_B^2$. Let us suppose that the sample is rectangular with dimensions L_x, L_y and that the left hand edge is at $x = -L_x$ and the right hand edge is at $x = 0$. Then the values of the wavevector k for which the basis state is substantially inside the sample run from $k = 0$ to $k = {L_x}/{l_B^2}$. It is clear that the states at the left edge and the right edge differ strongly in their k values and hence periodic boundary conditions are impossible.

The total number of states in each Landau level is then

$$N = \frac{L_y}{2\pi} \int_0^{L_x/l_B^2} dk = \frac{L_x L_y}{2\pi l_B^2} = N_\Phi$$

where

$$N_\Phi \equiv \frac{BL_x L_y}{\Phi_0}$$

is the number of flux quanta penetrating the sample. Thus there is one state per Landau level per flux quantum

Notice that even though the family of allowed wavevectors is only one-dimensional, we find that the degeneracy of each Landau level is extensive in the two-dimensional area. The reason for this is that the spacing between wave vectors allowed by the periodic boundary conditions $\Delta_k = {2\pi}/{L_y}$ decreases while the range of allowed wave vectors $\left[0, {L_x}/{l_B^2}\right]$ increases with increasing L.

The width of the harmonic oscillator wave functions in the nth Landau level is of order $\sqrt{n} l_B$. This is microscopic compared to the system size, but note that the spacing between the centers

$$\Delta = \Delta_k l_B^2 = \frac{2\pi l_B^2}{L_y}$$

is vastly smaller (assuming $L_y \gg l_B$). Thus the different basis states are strongly overlapping (but they are still orthogonal).

For simplicity we will restrict the remainder of our discussion to the lowest Landau level where the (correctly normalized) eigenfunctions in the Landau gauge are (dropping the index n = 0 from now on):

$$\psi_k(\vec{r}) = \frac{1}{\sqrt{\pi^{1/2} L l_B}} e^{-iky} e^{-\frac{1}{2l_B^2}(x+kl_B^2)^2}$$

and every state has the same energy eigenvalue $\epsilon_k = \frac{1}{2}\hbar\omega_c$.

We imagine that the magnetic field (and hence the Landau level splitting) is very large so that we can ignore higher Landau levels.

The expectation value of the current in the kth basis state is

$$\langle \vec{J} \rangle = -e\frac{1}{m}\langle \psi_k | (\vec{p} + e\vec{A}) | \psi_k \rangle$$

The y component of the current is

$$\begin{aligned}\langle J_y \rangle &= -\frac{e}{m\pi^{1/2} l_B}\int dx\, e^{-\frac{1}{2l_B^2}(x+kl_B^2)^2}(\hbar k + eBx)e^{-\frac{1}{2l_B^2}(x+kl_B^2)^2} \\ &= -\frac{e\omega_c}{\pi^{1/2} l_B}\int dx\, e^{-\frac{1}{l_B^2}(x+kl_B^2)^2}(x + kl_B^2)\end{aligned}$$

We see from the integrand that the current density is antisymmetric about the peak of the gaussian and hence the total current vanishes. This antisymmetry (positive vertical current on the left, negative vertical current on the right) is the remnant of the semiclassical circular motion.

Let us now consider the case of a uniform electric field pointing in the x direction and giving rise to the potential energy $V(\vec{r}) = eEx$

This still has translation symmetry in the y direction and so our Landau gauge choice is still the most convenient. Again separating variables we see that the solution is nearly the same as before, except that the displacement of the harmonic oscillator is slightly different. The Hamiltonian becomes

$$h_k = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2(x + kl_B^2)^2 + eEx$$

Completing the square we see that the oscillator is now centered at the new position

$$X_k = -kl_B^2 - \frac{eE}{m\omega_c^2}$$

and the energy eigenvalue is now linearly dependent on the particle's peak position X_k (and therefore linear in the y momentum)

$$\epsilon_k = \frac{1}{2}\hbar\omega_c + eEX_k + \frac{1}{2}m\bar{v}^2\ ,$$

where $\bar{v} = -{}^{E}/_{B}$. Because of the shift in the peak position of the wavefunction, the perfect antisymmetry of the current distribution is destroyed and there is a net current

$$\langle J_y \rangle = -e\bar{v}$$

This result can be derived either by explicitly doing the integral for the current or by noting that the wave packet group velocity is $\frac{1}{\hbar}\frac{\partial\epsilon_k}{\partial k} = \frac{eE}{\epsilon_k}\frac{\partial X_k}{\partial k}$=$\bar{v}$ independent of the value of k (since the electric field is a constant in this case, giving rise to a strictly linear potential).

It should be noted that the applied electric field 'tilts' the Landau levels in the sense that their energy is now linear in position. This means that there are degeneracies between different Landau level states because different kinetic energy can compensate different potential energy in the electric field. Nevertheless, we have found the exact eigenstates (i.e., the stationary states). It is not possible for an electron to decay into one of the other degenerate states because they have different canonical momenta. If however disorder or phonons are available to break translation symmetry, then these decays become allowed and dissipation can appear. The matrix elements for such processes are small if the electric field is weak because the degenerate states are widely separated spatially due to the small tilt of the Landau levels.