Magnetic monopole

Assume that there is a magnetic monopole with charge.
Similarly to the electric charge, the Gauss's law (for a closed manifold)

$$\int d\vec{S}\cdot\vec{B}=4\pi q_m$$

The $\vec{B}$ field is:

$$\vec{B}=q_m\frac{\vec{e}_r}{r^2}=q_m\frac{\vec{r}}{r^3}=q_m\frac{(x,y,z)}{(x^2+y^2+z^z)^{3/2}}$$

The vector potential $\nabla\times\vec{A}=\vec{B}$
The value of $\vec{A}$ is not unique, but they are all connected by a gauge transformation.
For example:

$$\vec{A}=q_m\frac{(y,-x,0)}{r(r-z)}$$

This vector potential has one problem. It is singular at the north pole $(z=r)$. In fact, one can prove that no matter which gauge one uses, there will always be a singular point.
This singularity is not a physical singularity. All physical observables are smooth and non-singular functions at this point. Only A (which is not a measurable quantity) shows singular behavior. In addition, the location of this singularity point is gauge dependent.
For example, using another gauge, which gives the same B field

$$\vec{A}=q_m\frac{(-y,x,0)}{r(r+z)}$$

This singular at the south pole $(z=-r)$.
Use the first one to describe the south hemisphere and the second one to describe the north hemisphere:

$$\vec{A}_N=q_m\frac{(-y,x,0)}{r(r+z)}\qquad\qquad\vec{A}_S=q_m\frac{(y,-x,0)}{r(r-z)}$$

At the equator, the vector potential is multivalued (depending on whether we use A_N or A_S. Because we know that A is not a physical observable and it is multivalued. As long as they differ by a gauge transformation, they describe the same physics (same B field).
The gauge transformation between A_N and A_S is:

$$\vec{A}_N=\vec{A}_S+2q_m\frac{(-y,x,0)}{(r-z)(r+z)}$$

At the equator $(z=0)$

$$\vec{A}_N=\vec{A}_S+2q_m\frac{(-y,x,0)}{r^2}=\vec{A}_S+2q_m\nabla\varphi$$

Gauge transformation:

$$\phi\to\phi'=\phi-\frac{\partial\Lambda(\vec{r},t)}{\partial t}$$
$$\vec{A}\to\vec{A}'=\vec{A}+\nabla\,\Lambda(\vec{r},t)$$
$$\psi(\vec{r},t)\to\psi'(\vec{r},t)=\psi(\vec{r},t)e^{i\frac{q_e}{\hbar}\Lambda(\vec{r},t)}$$

Here

$$\Lambda(\vec{r},t)=2q_m\varphi$$
$$\psi_N(\vec{r},t)=\psi_S(\vec{r},t)e^{i\frac{q_e}{\hbar}\Lambda(\vec{r},t)}=\psi_S(\vec{r},t)e^{i\frac{2q_eq_m}{\hbar}\varphi}$$

We know φ and $\varphi+2\pi$ are the same points, therefore

$$\frac{2q_e q_m}{\hbar} = n$$

where n is an integer.

$$\psi_N(\vec{r},t) = \psi_S(\vec{r},t)e^{in\varphi}$$

The magnetic charge is quantized (up till now we have used the $c = 1$ units)

$$q_m = \frac{c\hbar}{2q_e}n$$

For a closed surface enclosing a magnetic monopole, no matter what gauge one uses, the vector potential must have some singularities.

If A is a non-singular function on a closed manifold, the magnetic flux through this manifold must be zero.
To proof this, we cut the manifold into two parts D_I and D_{II}
The magnetic flux though D_I

$$\int_{D_I} d\vec{S}\cdot\vec{B} = \int_{D_I} d\vec{S}\cdot\nabla\times\vec{A} = \oint_{C_I} d\vec{l}\cdot\vec{A}$$

where C_I is the edge of D_I , and we used the Stokes' theorem for nonsingular functions.
The expression is the same for the magnetic flux though D_{II}
Therefore, the total magnetic field flus is

$$\int d\vec{S}\cdot\vec{B} = \oint_{C_I} d\vec{l}\cdot\vec{A} + \oint_{C_{II}} d\vec{l}\cdot\vec{A}$$

The edge of D_I and D_{II} are the same curve but their directions are opposite, therefore

$$\int d\vec{S}\cdot\vec{B} = 0$$

The only way to have nonzero magnetic flux here is to have some singular vector potential. If A is singular, we must use at least two different gauge choice to cover the whole manifold.
If we use A_I for D_I and A_{II} for D_{II}, we get

$$\int d\vec{S}\cdot\vec{B} = \oint_{C_I} d\vec{l}\cdot\left(\vec{A}_I - \vec{A}_{II}\right)$$

Replacing $\vec{A}_I = \vec{A}_N$ and $\vec{A}_{II} = \vec{A}_S$

$$\int d\vec{S}\cdot\vec{B} = \oint_{C_I} d\vec{l}\cdot\left(\vec{A}_N - \vec{A}_S\right) = 2q_m\oint_{C_I} d\vec{l}\cdot\nabla\varphi = 4\pi q_m$$

(Going from CGS to SI units $\vec{B} \to \vec{B}/_{4\pi}$)

Another choice of the vector potential, using spherical polar coordinates

$$A_\varphi^N = \frac{q_m}{r}\frac{1-\cos\theta}{\sin\theta}$$

$$A_\varphi^S = -\frac{q_m}{r}\frac{1+\cos\theta}{\sin\theta}$$

The gauge transformation connecting the two

$$A_\varphi^N = A_\varphi^S + \frac{2q_m}{r\sin\theta}\partial_\varphi\varphi$$

leads to the same result.