Properties of the superconducting condensate.

Simplest picture:

Those part of the free energy which depends directly on the condensate ($\psi(\vec{r})$) should depend on $F = F[\psi(\vec{r}), \nabla\psi(\vec{r})]$
In the case of charge less condensate (superfluid)

$$F = \int d\vec{r}\left(\frac{1}{2m^*}|-i\hbar\nabla\psi|^2 + a|\psi|^2 + b|\psi|^4\right)$$

In a charged system, the local $U(1)$ invariance requires $\vec{p} \to \vec{p} - e^*\vec{A}$,

$$F = \int d\vec{r}\left(\frac{1}{2m^*}\left|\left(-i\hbar\nabla - e^*\vec{A}\right)\psi\right|^2 + a|\psi|^2 + b|\psi|^4\right)$$

The gauge invariance means, that by the following gauge transformation
$\psi \to \tilde{\psi} = e^{i\chi(x)}\psi$, $A_\mu \to \tilde{A}_\mu - \frac{1}{e}\partial_\mu\chi(x)$
any measurable function should be invariant

$$f[\psi, A_\mu] = f[\tilde{\psi}, \tilde{A}_\mu]$$

Let $\psi(\vec{r}) = \sqrt{\rho(\vec{r})}e^{i\varphi(\vec{r})}$

$$F = \int d\vec{r}\left\{\frac{\hbar^2\rho}{2m^*}\left(\nabla\varphi - \frac{e^*}{\hbar}A\right)^2 + \frac{\hbar^2}{2m^*}(\nabla\rho)^2 + a\rho + b\rho^2\right\}$$

It is minimum if the first term is zero, i.e. $\nabla\varphi - \frac{e^*}{\hbar}A = 0$

$$F = \int d\vec{r}\left\{\frac{\hbar^2}{2m^*}(\nabla\rho)^2 + a\rho + b\rho^2\right\}$$

Homogenous case ($\nabla\rho = 0$)

$$F = \int d\vec{r}\{a\rho + b\rho^2\}$$

The minimum when $a > 0$ $\rho = 0$, $a < 0$, $\rho = \rho_0 = -\frac{a}{2b}$, in this case $a\rho + b\rho^2 = -\frac{a^2}{4b}$
therefore when $a < 0$ and $\rho = \rho_0$

$$F = \int d\vec{r}\left\{\frac{\hbar^2\rho_0}{2m^*}\left(\nabla\varphi - \frac{e^*}{\hbar}A\right)^2 - \frac{a^2}{4b}\right\}$$

It is minimum when $\nabla\varphi = \frac{e^*}{\hbar}\vec{A}$, taking the *curl* of both side

$$\nabla \times \nabla\,\varphi = 0 = \frac{e^*}{\hbar}\nabla \times \vec{A} = \frac{e^*}{\hbar}\vec{B}$$

No magnetic field inside a homogenous superconductor, that is the *Meissner* effect.

Superconducting ring:

$$\oint d\vec{r}\left(\nabla\varphi - \frac{e^*}{\hbar}\vec{A}\right) = \oint d\vec{r}\; 0 = 0$$

$$\oint d\vec{r}\,\nabla\varphi = \oint d\vec{r}\frac{e^*}{\hbar}\vec{A} = \frac{e^*}{\hbar}\int d\vec{r}\,\nabla \times \vec{A} = \frac{e^*}{\hbar}\int d\vec{r}\,\vec{B} = \frac{e^*}{\hbar}\Phi_B$$

Φ_B is the flux inside the superconducting ring.
Furthermore

$$\oint d\vec{r}\,\nabla\varphi = \varphi(\theta = 2\pi) - \varphi(\theta = 0) = 2\pi n$$

because ψ is a single valued function, therefore

$$\Phi_B = 2\pi n \frac{\hbar}{e^*} = \frac{nh}{2e} = n\Phi_0$$

where $\Phi_0 = \frac{h}{2e}$ the flux quantum. That is the *flux quantization*.

Electric current:

$$\vec{J} = \frac{e^*}{2m^*}\{\psi^*(-i\hbar\nabla - e^*\vec{A})\psi + \psi(-i\hbar\nabla - e^*\vec{A})\psi^*\} = \frac{e^*\rho}{m^*}(\hbar\nabla\varphi - e^*\vec{A})$$

Taking the *curl* of both sides

$$\nabla \times \vec{J} = -\frac{{e^*}^2\rho}{m^*}\vec{B}$$

Once more

$$\nabla \times \nabla \times \vec{J} = -\frac{{e^*}^2\rho}{m^*}\nabla \times \vec{B}$$

by the Maxwell equation $\nabla \times \vec{B} = \vec{J}$,
therefore

$$\nabla^2\vec{J} = \frac{\vec{J}}{\lambda_L^2}$$

$$\nabla^2\vec{B} = \frac{\vec{B}}{\lambda_L^2}$$

Where the London penetration length

$$\lambda_L = \sqrt{\frac{m^*}{{e^*}^2\rho}} = \frac{1}{\omega_{pl}} = \left(\frac{c}{\omega_{pl}}\right)$$

ω_{pl} is the superconductor plasma frequency (depend on the density of the superconducting electrons)

Around a half plane (on one side is superconductor, the other side is not superconductor)

$$B = B_0 e^{-x/\lambda_L}$$

and the same time

$$J = J_0 e^{-x/\lambda_L}$$

The superconducting current around the surface screens the external magnetic field.

Fluctuations:

$$F = \int d\vec{r} \left(\frac{1}{2m^*} \left| \left(-i\hbar\nabla - e^* \vec{A} \right) \psi \right|^2 + a|\psi|^2 + b|\psi|^4 \right)$$

Homogenous case, the minimum at $|\psi_0|^2 = -\frac{a}{2b}$, $(a < 0)$

Let

$$\psi = (\psi_0 + \delta\psi) e^{i(\varphi_0 + \delta\varphi)}$$

$\delta\psi$ describes the amplitude, $\delta\varphi$ the phase fluctuation of the condensate.

Inserting to F, and using the gauge transformation $\vec{A} \to \vec{A} + \frac{\hbar}{e^*}\nabla(\delta\varphi)$

$$F = \int d\vec{r} \left[\frac{\hbar^2}{2m^*} (\nabla\delta\psi)^2 + \frac{{e^*}^2}{2m^*} \vec{A}^2 |\psi|^2 - 2a(\delta\psi)^2 - \frac{{\psi_0}^2}{2} \right]$$

- The phase fluctuations were transformed out by the gauge transformation. No Goldstone mode in a charged condensed system.
- The photons have a finite mass: the photon term, because $|\psi|^2 \approx |\psi_0|^2 = \frac{n_s}{2}$, where n_s is the density of the superconducting electrons, $\frac{{e^*}^2 n_s}{2m^*} \frac{A^2}{2} = \omega_{pl}^2 \frac{A^2}{2}$

  We have already seen that $\omega_{pl}^2 = \frac{1}{\lambda_L^2} = \left(\frac{c}{\lambda_L}\right)^2$. Using the deBroglie relation $\lambda_L = \frac{\hbar}{m_A c} = \frac{c}{\omega_{pl}}$, therefore $m_A c^2 = \hbar\omega_{pl}$, m_A is the photon mass. The condensate gives a finite mass for the photons. This mass is quite universal, doesn't depend on the details (on the underlying microscopic theory). That is the *Anderson-Higgs mechanism.*
- The amplitude fluctuations have a finite mass $\mu \propto -a$. That is the *Anderson-Higgs boson.* The value of the mass of this boson depends on the details of the underlying microscopic model.

Lagrangian (minimal coupling)

The amplitude of the superconducting condensate is fixed by the GL theory

$$|\psi_0|^2 = -\frac{a}{2b}$$

The Lagrangian of the superconducting condensate could not depend on the phase itself because it is not gauge invariant. However, $\partial_\mu \varphi(r) - 2eA_\mu(r)/\hbar$ is gauge invariant. Therefore

$$L_S\left[\partial_\mu \varphi(r) - 2eA_\mu(r)/\hbar\right]$$

In addition, the total Lagrangian contains the electromagnetic field also, ($c = 1$)

$$L_{EM} = \frac{1}{2}(E^2 - B^2) = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

$$E_i = \left[\partial_t \vec{A} - \nabla A_0\right]_i = \partial_0 A_i - \partial_i A_0 = F_{0i}$$

$$B_i = \left(\nabla \times \vec{A}\right)_i = \epsilon_{ijk}\partial_j A_k = \frac{1}{2}\epsilon_{ijk}F_{jk}$$

The total Lagrangian ($2e = \hbar = 1$)

$$L = \frac{1}{2}\int d\vec{r}(E^2 - B^2) + L_S\left[\partial_\mu \varphi(r) - A_\mu(r)\right]$$

Charge and current

$$J_i = \frac{\delta L_S}{\delta A_i} = -\frac{\delta L_S}{\delta\, \partial_i \varphi}$$

$$\rho = \frac{\delta L_S}{\delta A_0} = -\frac{\delta L_S}{\delta\, \partial_0 \varphi}$$

Zero resistivity:

We know that the canonical momentum π is defined as

$$\pi = \frac{\delta L_S}{\delta\, \partial_0 \varphi}$$

So $\rho = -\pi$. Therefore the Hamiltonian should be considered a function of -ρ and φ, instead of $\partial_0 \varphi$ and φ.

The equation of motion is:

$$\partial_0 \varphi = \frac{\delta H_S}{\delta \pi} = -\frac{\delta H_S}{\delta \rho}$$

On the other hand, we know that

$$\frac{\delta H_S}{\delta \rho(r)} = V(r)$$

where V is the voltage. This is because $E = \int dr \rho(r)\, V(r)$

Therefore,

$$\partial_0 \varphi(r) = -V(r)$$

For a static system, the phase of the condensate should be time-independent, $\partial_0 \varphi = 0$. So we have $V(r) = 0$. This means that if we have a constant current in the system (the system is static), $V = 0$. No voltage but finite current. This is superconductivity.

The Josephson junction

Josephson junction has two superconductors separated by a thin layer of insulator. The Lagrange of a Josephson junction L_J depends on the phase difference between the two superconductors, which is also gauge invariant

$$L = L_{EM} + L_{S1} + L_{S2} + L_{junction}$$
$$L_{junction} = \mathcal{A}F(\Delta\varphi)$$

where $\mathcal{A}$ is the area of the junction. F$(\Delta\varphi)$ is a function of the phase difference between the two superconductors ($\Delta\varphi = \varphi_L - \varphi_R$). It is easy to notice that $F(\Delta\varphi)$ must be a periodic function with periodicity $\frac{2n\pi\hbar}{2e} = n\pi\hbar/e$. (Cooper pairs have charge $2e$)

$$F(\Delta\varphi) = F(\Delta\varphi + n\pi\hbar/e)$$

For a charge neutral particle,

$$\Delta\varphi = \int dr\, \nabla\varphi$$

However, for charged particles, in the presence of gauge field A, the gauge invariance $\Delta\varphi$ shall take the form

$$\Delta\varphi = \int dr\, (\nabla\varphi - A)$$

This formula is gauge invariant.
Therefore the current cross the junction is

$$J = \frac{\delta L_{junction}}{\delta A} = \mathcal{A}F'(\Delta\varphi)\frac{\delta\Delta\varphi}{\delta A} = -\mathcal{A}F'(\Delta\varphi)\frac{\delta A}{\delta A} = -\mathcal{A}F'(\Delta\varphi)$$

Now we apply a voltage V across the junction. Because we know that

$$\partial_0\, \varphi(r) = -V(r)$$

It is easy to notice that

$$\Delta\varphi = -Vt + constant$$

Therefore:

$$J = -\,\mathcal{A}F'(-Vt + constant)$$

By applying a fixed V, we found that the current is changing in time. Because F is a periodic function, F' is also a periodic function. So J is a periodic function of t, and the periodicity is

$$\pi\hbar/eV$$

Considering the action, and completely ignoring the fluctuations in the amplitude of the condensate,

$$S[A,\varphi]=\frac{\beta}{2}\int d^3r\,\left\{\frac{\rho_0}{m}\left(\nabla\varphi-\frac{2e}{\hbar}A\right)^2+(\nabla\times A)^2\right\}$$

In momentum space ($2e=\hbar=1$)

$S[A,\varphi]=\frac{\beta}{2}\sum_q\left\{\frac{\rho_0}{m}\left(i\vec{q}\varphi_{\vec{q}}-\vec{A}_{\vec{q}}\right)\left(-i\vec{q}\varphi_{-\vec{q}}-\vec{A}_{-\vec{q}}\right)+\left(\vec{q}\times\vec{A}_{\vec{q}}\right)\left(-\vec{q}\times\vec{A}_{-\vec{q}}\right)\right\}=$

$\frac{\beta}{2}\sum_q\left\{\frac{\rho_0}{m}\left[q^2\varphi_{\vec{q}}\varphi_{-\vec{q}}-2i\vec{q}\vec{A}_{-\vec{q}}\varphi_{\vec{q}}+\vec{A}_{\vec{q}}\vec{A}_{-\vec{q}}\right]+\left(\vec{q}\times\vec{A}_{\vec{q}}\right)\left(-\vec{q}\times\vec{A}_{-\vec{q}}\right)\right\}$

Performing a gaussian integral on the φ degrees of freedom $\left(\int e^{-x^2+xy}\sim e^{y^2}\right)$

Or using the gaussian integral

$$\int_{-\infty}^{\infty}dx e^{-\frac{1}{2}ax^2+bx}=e^{\frac{b^2}{2a}}\sqrt{\frac{2\pi}{a}}$$

$$S[A]=\frac{\beta}{2}\sum_q\left\{\frac{\rho_0}{m}\left[\vec{A}_{\vec{q}}\vec{A}_{-\vec{q}}-\frac{\left(\vec{q}\vec{A}_{\vec{q}}\right)\left(\vec{q}\vec{A}_{-\vec{q}}\right)}{q^2}\right]+\left(\vec{q}\times\vec{A}_{\vec{q}}\right)\left(-\vec{q}\times\vec{A}_{-\vec{q}}\right)\right\}=$$

Break up $\vec{A}$ into a longitudinal and a transverse part: $\vec{A}_{\vec{q}}=\vec{A}_{\vec{q}}-\frac{\vec{q}\left(\vec{q}\vec{A}_{\vec{q}}\right)}{q^2}+\frac{\vec{q}\left(\vec{q}\vec{A}_{\vec{q}}\right)}{q^2}$

The first two terms are the transverse, the last is the longitudinal part.

Only the transverse part will contribute to the magnetic field, since $\vec{B}_{\vec{q}}=\vec{q}\times\vec{A}_{\vec{q}}^{\perp}$.

$$S[A]=\frac{\beta}{2}\sum_q\left(\frac{\rho_0}{m}+q^2\right)\vec{A}_{\vec{q}}^{\perp}.\vec{A}_{-\vec{q}}^{\perp}.$$

The equation of motion

$$\left(\frac{\rho_0}{m}-\nabla^2\right)A_\perp=0$$

That can be summarized as:

The Goldstone soft mode φ, due to the coupling between the Goldstone mode φ and the gauge field A, upon integrating the Goldstone mode, the gauge field acquires a mass.

The photon (vector potential) field has eaten up the Goldstone mode to become fat.

Originally we had three degrees of freedom, one from the Goldstone mode two from the gauge field (without mass the gauge field is pure transverse mode). Now we have three modes again, the gauge field is now not purely transverse but it gained a longitudinal component.

The frequency of the transverse mode (now restoring e)

$$\omega_q^{\perp}=\sqrt{\frac{e^2n_S}{m}+q^2}$$

Using a more detailed analysis the frequency of the longitudinal mode

$$\omega_q^{\parallel} = \sqrt{\frac{e^2 n_S}{m} + q^2 \frac{n_S}{b}}$$

GL equation

The free energy of the superconducting state by GL is (using the notation of the Sólyom book), measuring from the free energy of the normal state:

$$F = \int d\vec{r}\left(\frac{1}{2m^*}\left|\left(-i\hbar\nabla - e^*\vec{A}\right)\psi\right|^2 + \alpha(T)|\psi|^2 + \frac{1}{2}\beta(T)|\psi|^4 + \frac{1}{2}B^2\right)$$

Here the magnetic induction $\vec{B}$ is fixed (Helmholtz free energy)

The saddle point equations

$$\frac{\partial F}{\partial \psi^*} = 0 \implies \frac{1}{2m^*}\left(-i\hbar\nabla - e^*\vec{A}\right)^2\psi + \alpha(T)\psi + \beta(T)\psi|\psi|^2 = 0$$

$$\frac{\partial F}{\partial A} = 0 \implies \frac{i\hbar e^*}{2m^*}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right] - \frac{e^{*2}}{m^*}\vec{A}|\psi|^2 - \vec{\nabla}\times\vec{B} = 0$$

By the Maxwell equation $\vec{\nabla}\times\vec{B} = \vec{J}$

$$\vec{J} = \frac{i\hbar e^*}{2m^*}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right] - \frac{e^{*2}}{m^*}\vec{A}|\psi|^2 = \frac{e^*|\psi|^2}{m^*}\left(\hbar\vec{\nabla}\varphi - e^*\vec{A}\right)$$

These are the celebrated Ginzburg-Landau equations.

The boundary condition on the surface

$\vec{n}\left(-i\hbar\nabla - e^*\vec{A}\right)\psi = 0$ or a pure imaginary constant.

The uniform solution, without electromagnetic field

$$\left(\alpha(T) + \beta(T)\right)|\psi_0|^2 = 0 \implies \psi_0 = \sqrt{\frac{|\alpha|}{\beta}}e^{i\varphi} \equiv \sqrt{n_S}e^{i\varphi}$$

Inserting this value of the order parameter the free energy density difference between the superconducting and the normal state, i.e. the condensation energy density

$$f_S - f_N = -\frac{|\alpha(T)|^2}{2\beta} \equiv -\frac{\mu_0 H_c^2}{2}$$

H_c is the thermodynamic critical field.

The interface of a SC with a normal material. In the absence of fields and currents and taking a uniform solution parallel to the interface i.e. $\psi(\vec{r}) = \psi(\vec{x})$

$$-\frac{\hbar^2}{2m^*}\psi'' + \alpha\,\psi + \beta\psi^3 = 0$$

Defining $f = {}^{\psi}/_{\psi_0}$

$$\frac{\hbar^2}{2m^*|\alpha|}f'' + f - f^3 = 0$$

The combination $\xi^2 = \frac{\hbar^2}{2m^*|\alpha|}$ has units of $[length]^2$. It is called the GL coherence length and gives he typical scale for the variation of ψ. To demonstrate that, deep in the bulk $f = 1$, define $g(x)$ to be variations from this ideal value $f(x) = 1 + g(x)$. Assuming $g(x) \ll 1$, we can linearize the above equation $g'' = \frac{2}{\xi^2}g \implies g(x) \sim e^{\pm\sqrt{2}x/\xi}$

Which shows that small disturbances in f decay over the length scale ξ.

We have identified two length scales : the magnetic field penetration depth $\lambda_L = \sqrt{\frac{m^*}{e^{*2}|\psi_0|^2}} = \sqrt{\frac{m^*}{e^{*2}n_S}} = \sqrt{\frac{m^*\beta(T)}{e^{*2}|\alpha(T)|}}$ and the coherence length $\xi = \sqrt{\frac{\hbar^2}{2m^*|\alpha(T)|}}$

We naturally expect that the dimensionless ratio of these lengths will be an important parameter of the theory.

$$F = \int d\vec{r}\left(\frac{1}{2m^*}\left|\left(-i\hbar\nabla - e^*\vec{A}\right)\psi\right|^2 + \alpha(T)|\psi|^2 + \frac{1}{2}\beta(T)|\psi|^4 + \frac{1}{2}B^2\right)$$

Rescale all the dimension full parameters in order to obtain equations in terms of dimensionless parameters only.

Rescale:

$$x = \xi\tilde{x}\,, \psi = \sqrt{n_S}\tilde{\psi}\,, \vec{A} = \frac{\hbar}{e^*\lambda_L}\vec{a} \equiv \frac{\Phi_0}{2\pi\lambda_L}\vec{a}, n_S = -\frac{\alpha(T)}{\beta(T)}$$

Where $\Phi_0 = \frac{h}{e^*}$ is the natural unit of magnetic flux called a flux quanta.

Now writing the free energy in terms of the rescaled quantities we have

$$F = \xi^d n_S|\alpha| \int d^d\tilde{x}\left\{\left|\left(\vec{\nabla} - \frac{i}{\kappa}\vec{a}\right)\tilde{\psi}\right|^2 - \left|\tilde{\psi}\right|^2 + \frac{1}{2}\left|\tilde{\psi}\right|^4 + \left(\vec{\nabla}\times\vec{a}\right)^2\right\}$$

Where $\kappa = {}^{\lambda_L}/_{\xi}$, and $n_S|\alpha| = \mu_0 H_c^2$

Remarkably the behavior of the system (in the saddle point) is controlled by a single dimensionless parameter κ called the GL parameter.

${}^{1}/_{\kappa}$ is the dimensionless coupling constant to the electromagnetic field.

$${}^{1}/_{\kappa} = \frac{e^*\hbar}{m^*\sqrt{2\beta(T)}}$$

The first GL equation in dimensionless form

$$\left(\vec{\nabla} - \frac{i}{\kappa}\vec{a}\right)^2\tilde{\psi} + \tilde{\psi} - \tilde{\psi}\left|\tilde{\psi}\right|^2 = 0$$

Vortices in type II superconductors

We have seen that the long wavelength phase fluctuations, the Goldstone modes are absent. What about the short wavelength phase fluctuations, or shortrange distortion states of the phase?

$\kappa \gg 1$. In this case we can take $|\psi|^2 \approx n_S = const$ while keeping in mind that this theory has a short length cutoff ξ. Now we have only the second GL equation

$$\vec{J} = \frac{1}{\lambda_L^2}\left(\frac{\Phi_0}{2\pi}\vec{\nabla}\varphi - \vec{A}\right)$$

or in another form (Maxwell equation $\vec{J} = \vec{\nabla} \times \vec{B}$)

$$\frac{\Phi_0}{2\pi}\vec{\nabla}\varphi = \vec{A} + \lambda_L^2 \vec{\nabla} \times \vec{B}$$

Let $\varphi(\vec{r}, \vartheta) = \vartheta$ in polar coordinates, this means $\vec{\nabla}\varphi = \frac{1}{r}\hat{\vartheta}$

$\vec{\nabla} \times \vec{\nabla}\varphi = \vec{\nabla} \times \left(\frac{1}{r}\hat{\vartheta}\right) = 2\pi\hat{z}\delta^2(r)$ because $\vec{\nabla} \times \left(\frac{1}{r}\hat{\vartheta}\right) = 0$ for $r > 0$ and $\int dS\,\vec{\nabla} \times \vec{\nabla}\varphi = \oint d\vec{l}\;\vec{\nabla}\varphi = 2\pi$

We know that the short distance cutoff in this model is ξ but in case $\lambda_L \gg \xi$ we can use this approximation.

Taking the curl of the above equation

$$\Phi_0\hat{z}\delta^2(r) = \vec{B} + \lambda_L^2\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{B} - \lambda_L^2\nabla^2\vec{B}$$

The exact solution of this equation

$$B_z(r) = \frac{\Phi_0}{2\pi\lambda_L^2}K_0\left(\frac{r}{\lambda_L}\right)$$

where K is the zeroth-order modified Bessel function. Inside the vortex core

$$B_z(r \leq \xi) = \frac{\Phi_0}{2\pi\lambda_L^2}K_0\left(\frac{\xi}{\lambda_L}\right)$$

Limiting forms of this function

$B_z(r) \approx \frac{\Phi_0}{2\pi\lambda_L^2}\left(ln\left(\frac{\lambda_L}{r}\right) + 0.12\right)$ if $\xi \ll r \ll \lambda_L$

$B_z(r) \approx \frac{\Phi_0}{2\pi\lambda_L^2}\sqrt{\frac{\pi\lambda_L}{2r}}\,e^{-r/\lambda_L}$ if $r \gg \lambda_L$