# TKKN formula

(Thouless, Kohmoto, Nightingale, den Nijs, 1982)

Starting from the Kubo formula of conductivity based on the linear response theory. We consider a Hamiltonian $H = H_0\left(k+\frac{e}{\hbar}A\right) \sim H_0(k) + j \cdot A = H_0(k) + H_1$, where $H_0(k)$ is the unperturbed Hamiltonian and $H_1$ is the perturbation from external field. In the dc limit, the conductivity tensor is given by

$$\sigma_{\alpha\beta} = i\hbar \sum_{nm} \left(n_F(\epsilon_n) - n_F(\epsilon_m)\right) \frac{(j_\alpha)_{nm}(j_\alpha)_{mn}}{(\epsilon_n - \epsilon_m)^2}$$

$$= i\hbar \sum_{nm} n_F(\epsilon_n) \frac{(j_\alpha)_{nm}(j_\alpha)_{mn} - (j_\beta)_{nm}(j_\alpha)_{mn}}{(\epsilon_n - \epsilon_m)^2}$$

where $n_F$ is the Fermi distribution function, $j$ is the current operator $j_\alpha = \frac{e}{\hbar}\frac{\partial H_0}{\partial k_\alpha}$ and $(j_\alpha)_{nm} = \langle u_n | j_\alpha | u_m \rangle$. Here $\epsilon_n$ and $|u_n\rangle$ are the eigen-energy and eigen wavefunction of unperturbed Hamiltonian $H_0$ .

Since $(j_\alpha)_{nm} = \langle u_n | j_\alpha | u_m \rangle = \frac{e}{h}\left\langle u_n \middle| \frac{\partial H_0}{\partial k_\alpha} \middle| u_m \right\rangle = \frac{e}{h}(\epsilon_n - \epsilon_m)\left\langle \frac{\partial u_n}{\partial k_\alpha} \middle| u_m \right\rangle$, the formula for Hall conductivity can be simplified as

$$\sigma_{xy} = \frac{ie^2}{\hbar} \sum_n n_F(\epsilon_n) \left( \left\langle \frac{\partial u_n}{\partial k_x} \middle| \frac{\partial u_n}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_n}{\partial k_y} \middle| \frac{\partial u_n}{\partial k_x} \right\rangle \right)$$

Here the summation over $n$ include the integral over the momentum $k$. Consider only on zero temperature, the formula is rewritten as

$$\sigma_{xy} = \frac{ie^2}{\hbar} \int \frac{d^2k}{(2\pi)^2} \sum_{n\,occ} \left( \left\langle \frac{\partial u_n}{\partial k_x} \middle| \frac{\partial u_n}{\partial k_y} \right\rangle - \left\langle \frac{\partial u_n}{\partial k_y} \middle| \frac{\partial u_n}{\partial k_x} \right\rangle \right)$$

In which the summation over $n$is just for all the occupied band.

In terms of the Berry curvature $\mathcal{F}_n^{ij}(\vec{k}\,)$

$$\sigma_{xy} = \frac{e^2}{\hbar} \int \frac{d^2k}{(2\pi)^2} \sum_{n\,occ} \mathcal{F}_n^{xy}(\vec{k}\,)$$