Topology

In geometry, if a manifold M_1 can be adiabatically deformed into M_2, they have the same topology. Otherwise they are topologically different. They are characterized by a topological index. It is an integer number.

For a 2D closed manifold, for each point of the manifold we can fit different circles. The inverse radius $\kappa = {}^{1}/_{R}$ gives the curvature. Among the curvatures, the largest and the smallest are the principal curvatures κ_1 and κ_2. The Gaussian curvature is $K = \kappa_1 \kappa_2$
For a closed 2D manifold

$$\frac{1}{2\pi} \oiint_M dS \ K = \chi_M$$

Gauss-Bonnet theorem: χ_M always an integer.
For orientable closed manifolds $\chi_M$can only be even integers. Orientable means that we can distinguish two sides of the surface. (Möbius strip is not orientable)
Sphere: $\chi_M = 2$
Torus: $\chi_M = 0$
Double torus: $\chi_M = -2$
And so on.

This topological index is integer, only if we are considering a closed manifold which has no boundary.

The genus of the manifold

$$g = 1 - \frac{\chi_M}{2}$$

The genus measure the number of “handles” on a object. A sphere has no Handel, so $g = 0$, for torus $g = 1$, …….

For polyhedrons

$$\chi_M = V - E + F$$

where V, E, F are the number of vertices (corners), edges and faces respectively.

From mathematical point of view, the following three objects are the same thing: Gaussian curvature of K, the magnetic field B, and the Berry curvature $\mathcal{F}$.

$\oiint_M dS \ K = 2\pi\chi_M$ quantized: χ_M is an integer
Known as the Euler characteristic, which measures the topological nature of the manifold M
$\oiint_M dS \ B = \oiint_M dS \ B_n = \frac{c\hbar}{2q_e} n$ quantized: n is an integer
Known as magnetic charge, which measures the number of magnetic monopole inside M

$\oiint_{BZ} d\vec{k} \ \mathcal{F} = 2\pi C$ quantized: C is an integer, known as the TKKN invariant or the Chern number.
which measures the quantized Hall conductivity for a topological insulator